

Constructing a Square with the Area $1/n$ of a Given Square

Dikshha Sinha

We present here a solution to the problem posed on page 26

A construction to halve the area of a given square.

In the following figure let $ABCD$ be a given square (of unknown side length). Let $PB = BS = 1$ unit, and $BQ = 2$ units. Let PRQ be a semicircle passing through points P and Q . Then the area of the square $XYZB$ must be half of that of $ABCD$.

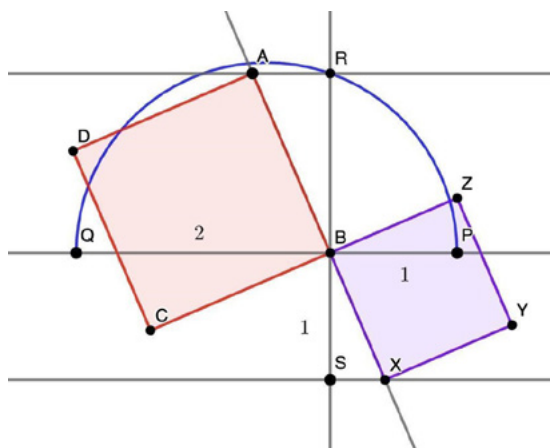


Figure 1

This question and this picture were submitted by one of our authors and it made us think. Do spend some time observing this picture – Is the area of square $XYZB$ half the area of the given square $ABCD$? If so, why? We explain below.

If we apply computational thinking to solve this problem, this is the first step:

Statement of the Problem: Is the area of square $XYZB$ half the area of the given square $ABCD$? If so, why?

Let us try to decompose the steps: We are given that $QB = 2$ units, and $PB = BS = 1$ unit.

Given a square whose area is x square units, we want to draw a square whose area is $\frac{x}{2}$ square units. A quick observation reveals that this is the same as given a line segment of \sqrt{x} units, we want to construct a line segment of $\sqrt{\frac{x}{2}}$ units.

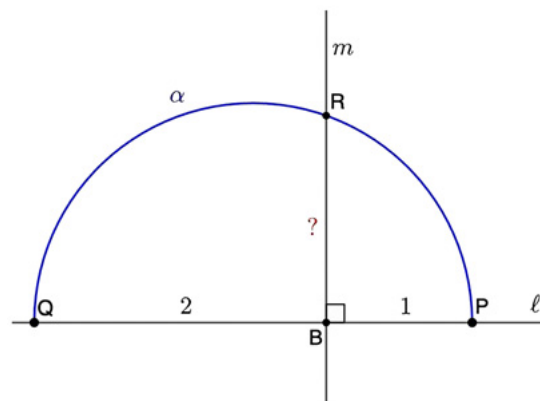


Figure 2

Keywords: Construction, fractional areas, reasoning

Step 1. Mark three points P , B and Q on a line l , such that B lies in between P and Q . $QB = 2$ units and $PB = 1$ unit.

Step 2. We draw a semicircle α with PQ as the diameter.

Step 3. Let the line m perpendicular to l and passing through B intersect α at R . (See Figure 2)

What will BR be?

From the right-angled $\triangle PBR$, $PR^2 = PB^2 + RB^2$.

From the right-angled $\triangle RBQ$, $QR^2 = RB^2 + BQ^2$.

From the right-angled $\triangle PRQ$, $PQ^2 = PR^2 + RQ^2$ (note that $\angle PRQ$ is a right angle in the semicircle).

By using these three equations, we arrive at the fact that $BR = \sqrt{2}$.

Step 4. Mark the point S on the line m at the other side of l such that $BS = 1$ unit (see Figure 3)

Step 5. Draw the line α parallel to l passing through R , and the line b parallel to l passing through S .

Step 6. Choose any point A on the line α and join the points B and A via the line c . Let c intersect b at X .

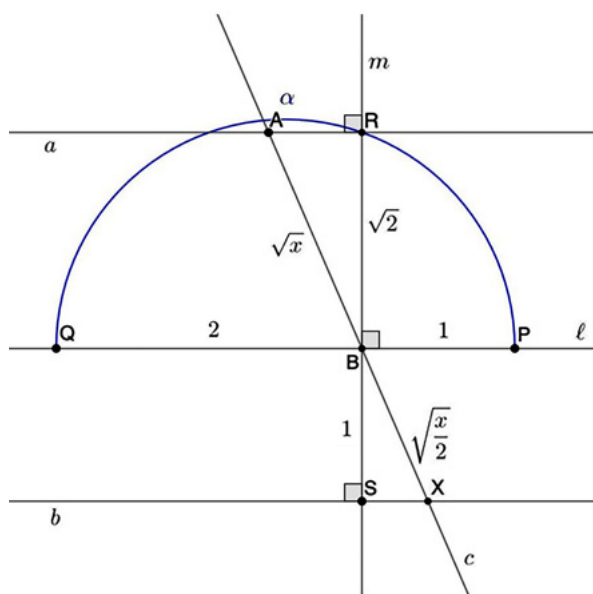


Figure 3

Now $\triangle ARB$ and $\triangle XBS$ are similar. So, if $AB = \sqrt{x}$ units, then $BX = \sqrt{\frac{x}{2}}$ units.

Thus, if we construct a square with AB and XB as respective side lengths, then necessarily we should have that the square with AB as a side, must be two times the area of the square with XB as a side. Thus, by decomposing the solution steps, we conclude that the area of square $XYZB$ is indeed half the area of the given square $ABCD$.

The reader might have already observed that there are simpler constructions of squares that can halve the area of a given square, such as in Figure 4. Here $ABCD$ is a given square, and P , Q , R and S are midpoints of sides AB , BC , CD and DA respectively. However, the above construction can be extended to construct a square with area $\frac{1}{n}$ of a given square, by taking $BQ = n$ instead of 2 (See Problem 1 below).

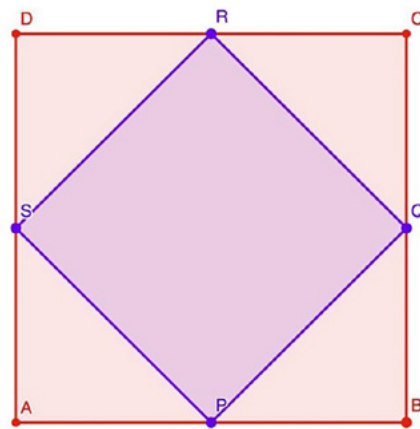


Figure 4

Why stop at halving the area of a square? We claim that this construction can do more! We provide a set of problems for the reader to try their hands on.

Problems

1. If we assume that $BQ = n$ units, what would be the length of BR ? What would be the length of BX ? What is the ratio between the areas of $ABCD$ and $XYZB$?
2. The above construction works only if the side AB is more than or equal to BR . What if $AB < BR$? The above construction can be

slightly modified to accommodate this possibility (Figure 5). Argue why the following construction would work. Here β is a semicircle joining the points B and R , and γ is the semicircle joining the points B and S .

3. Do the above constructions work if we want to halve the area of an equilateral triangle? A regular hexagon? A regular 13-gon? A circle?
4. Given a two dimensional shape, can you argue if the above constructions give a shape which is similar to the given shape and has area $1/n$ of that of the given shape?

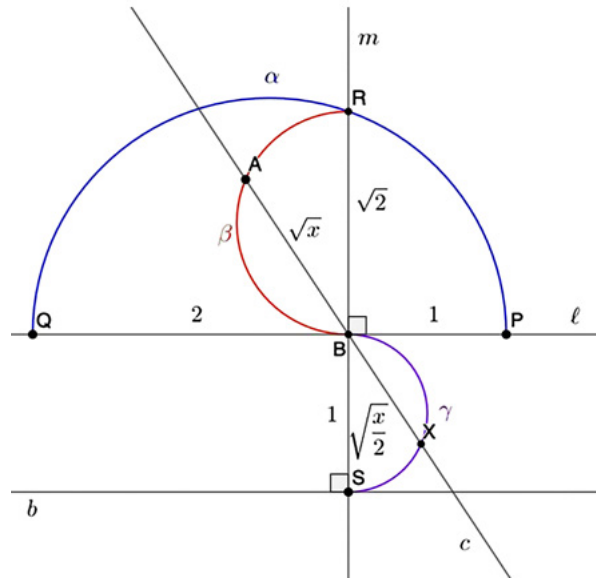


Figure 5



DIKSHHA SINHA is a student of class X at Podar International School CIE Nerul Navi Mumbai. Her passion and love for mathematics has led her to participating actively in Raising A Mathematician Training Program by RAM Foundation and MathPath at USA. Dikshha may be contacted at dictionarycube@gmail.com