Student Corner

Exploring and Connecting: Inradius and Factorisation

SWASTI PATIL,
SIDDHARTH LAXSHMISHA,
DAVIN CHRISTINO,
with comments by
VINAY NAIR

During a few online sessions with students an exploration was made on inradius and its relationship with integer-sided right triangles. This article compiled by 9-year-old Swasti, explains the thought process that evolved during the sessions. In a regular classroom setting, a problem on calculating the inradius for a particular right triangle may be tackled and considered solved when the required value is obtained. But in these sessions, the goal was not only to arrive at answers but also to pose new problems that came to mind during the process of solution. Thus, you will see how the problem did not end with finding the inradius and how the students probed (and were sometimes nudged or even pushed) into questions involving how many possible solutions exist. In this process, they wandered into a factorization problem through a problem in geometry.

This is an account of a discussion with students of different age groups (9 to 14 years). All of them had the understanding of congruence, tests of congruence in triangles, inradius, generalisation of Pythagorean triplets, some properties of circles – such as the radius and tangent are perpendicular at the point of contact, solving equations and manipulating algebraic expressions.

Keywords: Inradius, Right angled triangle, Pythagorean Triplets, Exploration

Question: Can we find how many right triangles with integer side lengths exist for a given integer inradius?

This question was an outcome of an exploration in Pythagorean triplets that further led to finding the inradius of a right-triangle whose sides are integers. While observing the list of such Pythagorean triplets along with their inradii, it was seen that for a given inradius, there could be more than one integer-sided right triangle. It was here that the teacher pushed the class into exploring the question stated above (without the teacher himself knowing that it would lead to a factorisation problem).

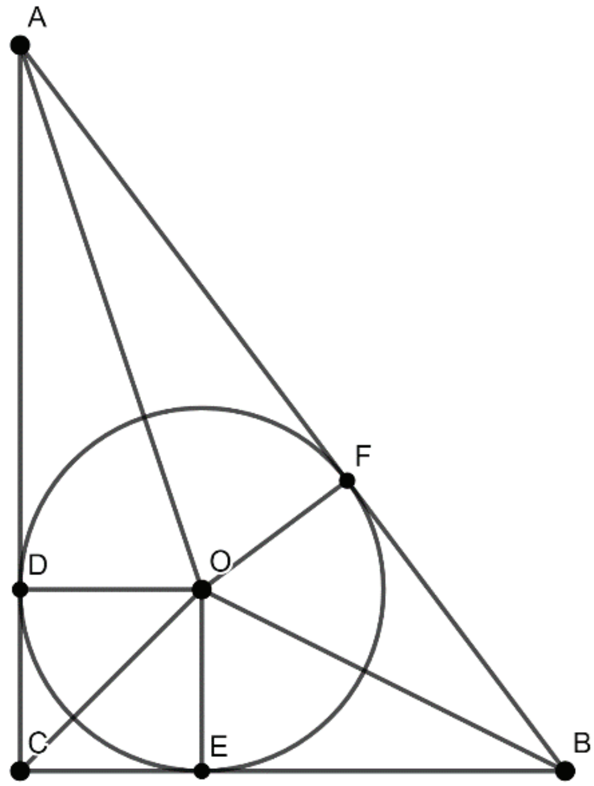

Figure 1.

In Figure 1, ΔABC is a triangle right angled at C. Point O is the incentre of ΔABC.

We started with the above question which led us to an exploration and we arrived at something interesting with respect to the inradius of a right triangle with integer side lengths.

The well-known formula $[\Delta ABC] = \frac{1}{2} \cdot r \cdot (AB + BC + CA)$ can be further simplified to get the value of r as $\frac{BC \cdot AC}{AB + BC + CA}$. Here $[\Delta ABC]$ refers to the area of the triangle ABC.

We use this formula and explore further.

Let D, E and F be the points where the incircle meets the sides AC, CB and BA, respectively.

Let $\overline{BC} = a$, $\overline{AC} = b$, and $\overline{AB} = c$. [(a, b, c) is a Pythagorean Triplet]

We know that Pythagorean triplets can be found using *Brahmagupta's formula*, i.e., $\left(m^2 + n^2, m^2 - n^2, 2mn\right)$, which generates *Pythagorean Triplets* (here m and n are natural numbers such that $m > n$). [*Comment.* By imposing additional conditions (that m and n are coprime and have opposite parity), we can ensure that the Pythagorean triplet is primitive, i.e., that the numbers in the triplet are coprime. However, we do not give the proof of this statement in this article, for brevity.]

The table below lists Pythagorean Triplets generated using m and n such that $m = n + 1$ and the corresponding inradii computed using the above formula. Here (a, b, c) is a Pythagorean Triplet such that $a = m^2 - n^2$, $b = 2mn$ and $c = m^2 + n^2$.

m	n	a	b	c	inradius
2	1	3	4	5	1
3	2	5	12	13	2
4	3	7	24	25	3
5	4	9	40	41	4
6	5	11	60	61	5

Observation 1. When $m - n = 1$, the inradius is n.

Let us observe the inradius when $m - n > 1$.

m	n	a	b	c	inradius
3	1	8	6	10	2
4	2	12	16	20	4
5	2	21	20	29	6
5	3	16	30	34	6
6	3	27	36	45	9

Observation 2. When $m - n = 2$, the inradius is $2n$.

Claim (based on the observations; this claim was made by a student). The inradius is $n\,(m - n)$.

Proof:

In polygon $CDOE$,

$OD = OE$ (radii of the same circle)

$\measuredangle D = \measuredangle O = \measuredangle C = \measuredangle E = 90^\circ$.

$\therefore$ Polygon $CDOE$ is a square.

$\therefore CD = CE = OE = OD = r$.

In $\Delta AOD \& \Delta AOF$,

$$\overline{AO} = \overline{AO}$$
$$\measuredangle ADO = \measuredangle AFO = 90^\circ$$
$$OD = OF$$

$\therefore$ By RHS,

$$\Delta AOD \cong \Delta AOF$$
$$\implies \overline{AD} = \overline{AF}$$

Let side $\overline{AD} = \overline{AF} = q$.

Similarly, we can also show that $\Delta FOB \cong \Delta EOB \implies FB = EB$.

$$\therefore \text{Let } FB = EB = p.$$

We obtain the following equations:

$$r + q = \overline{AC} = b$$
$$p + r = \overline{BC} = a$$
$$q + p = \overline{AB} = c$$

We need to find r.

Adding the 3 equations, we get

$$2p + 2q + 2r = a + b + c$$
$$\implies p + q + r = \frac{a+b+c}{2}$$
$$\implies r = \frac{a+b+c}{2} - (p+q) = \frac{a+b+c}{2} - c = \frac{a+b-c}{2}$$

$\therefore$ The inradius of a right-angled triangle with hypotenuse c and sides a and b is $\frac{a+b-c}{2}$.

Substituting $a = m^2 - n^2, b = 2mn$ and $c = m^2 + n^2$:

$$r = \frac{\left(m^2 - n^2\right) + (2mn) - \left(m^2 + n^2\right)}{2} = \frac{2mn - 2n^2}{2} = mn - n^2 = n(m-n).$$

Hence proved. Using this result, we can easily prove Observation 1 & Observation 2.

We can get a more general result by multiplying $m^2 + n^2$, 2mn, $m^2 - n^2$ by a constant k (it should be a positive integer). This ensures that all Pythagorean Triplets have been covered. The inradius of the same will now be $kn(m-n)$.

Finally, we get to the problem of how many right triangles exist for a given inradius where the side lengths and inradius are integers.

We know that $r = kn(m-n)$, hence the problem reduces to factorisation. [*Comment.* The students did not use the word 'factorisation' explicitly, but they said – we need to find the factors of ' r ' to find all possible side lengths.]

Let us look at an example. When the inradius is 4, possible values for $(m - n)$ and n are:

k	n	m − n	m	a	b	c
1	1	4	5	24	10	26
1	2	2	4	12	16	20
1	4	1	5	9	40	41
2	2	1	3	10	24	26
2	1	2	3	16	12	20
4	1	1	2	12	16	20

If we discard repetitions, the possible triangles will be

1. (24, 10, 26)
2. (12, 16, 20)
3. (9, 40, 41).

Here's another way to proceed:

We know that the inradius $r = \dfrac{a+b-c}{2}$.

$$\therefore 2r = a + b - c$$

$$\Rightarrow 2r - a - b = -c = -\sqrt{a^2 + b^2}$$

Squaring both the sides, we get

$$a^2 + b^2 = (2r - a - b)^2 = 4r^2 + a^2 + b^2 + 2ab - 4ar - 4br$$

$$\Longrightarrow 2r^2 = 4r^2 - 2ar - 2br + ab = (a - 2r)(b - 2r)$$

This again reduces to a factorisation problem.

In this entire exploration, we were intrigued by the way exploration with Pythagorean triplets and inradius led us to a factorisation problem. We also saw that there are multiple ways to compute the inradius of an integer-sided right triangle.

Takeaways from a teacher's perspective

1. Usually, teachers are well prepared for a session and the aim is to enable the students to learn concepts and apply them in solving problems. In my experience of working with students who are passionate about mathematics and willing to go that extra mile in exploration, going to a class with a good question that I myself hadn't explored, also helped in multiple ways. The first advantage is that I don't end up steering them to a desired goal and that helps in bringing out their creativity. As we don't have a chalked-out path to walk, we ask ourselves and each other – what is the next good question to explore? This way, new questions come up from the students' side. That is the second advantage. Slowly, this becomes a thinking habit. Thus, whenever they see a problem, they start to think of more problems that can be constructed from the given problem, whether a generalisation would be possible, and so on.

2. While working on the problem in this article, when they tried to find all possible side lengths of a right triangle with integer side lengths, they found out that some solutions weren't getting generated by the factorisation of $r = n(m - n)$ and that made them realise that not all Pythagorean Triplets can be generated using Brahmagupta's formula. The first example that they hit was 9, 12, 15 where the inradius was 3, but for inradius 3, while considering $n(m - n)$, the triangle they obtained was 7, 24, 25. This reminded them of the fact that Brahmagupta's expressions generate only Primitive Pythagorean Triplets. After that, multiplying Brahmagupta's expressions by a new variable 'k' was a natural process because Brahmagupta's expressions when multiplied by another natural number k, give all possible Pythagorean Triplets.

3. Normally, factorisation is taught as a separate topic and many students don't find any purpose in learning or applying the ways to factorise an algebraic expression. However, in a problem like this, they started with a geometry problem, did some smart algebraic manipulation and ended up in factorisation. Such explorations result in getting into a decompartmentalised way of looking at mathematics.

SWASTI PATIL is a 9 year old unschooled girl from Pune. Apart from Mathematics and Coding, she also has a love for theatre, for plants, for reading, and singing. She also plays the instrument *jaltarang*.

SIDDHARTH LAXMISHA is a 5th grader from The Study L'école Internationale, Puducherry. He loves Mathematics, Physics, Coding, playing the Keyboard and most of all, eating french fries.

DAVIN CHRISTINO is a 10th grader from Good Earth School, Chennai. He loves Physics, Math, Coding, Chemistry and playing Football.

VINAY NAIR works with all three of them (and others like them who are precocious in mathematics) in a course, *Ganit Manthan*, where they together explore problems, create and prove/disprove conjectures. The focus of the sessions is not just to solve problems but also to pose interesting problems that the students come up with. He can be reached at nairvinayr@gmail.com