The Bucket Game: Strategy, Proof, and Generalization

Article contributed by Aarav Lad, Akshobhyan R., Hari Annamalai, Nidhi Nayak, and Shaunak Okhde

1 The Problem

Alice and Bob are playing a game. There are 7 white counters and 3 black counters in a bucket. On each turn, a player must do exactly one of the following:

- Either remove between 1 and 3 white counters; or
- Remove 1 or 2 black counters.

Players alternate turns. **Whoever removes the last counter from the bucket wins**. Alice moves first.

> **Question.** What should Alice's first move be, so that she is guaranteed to win no matter how Bob plays?

A. 1 white B. 2 white C. 3 white D. 1 black E. 2 black

This looks like a self-contained puzzle. But the idea behind the solution scales up to a game with any number of counters and any removal limits. We will build up the solution in stages.

2 What Happens With a Single Pile of Counters?

Before mixing two colours together, it helps to understand how a single pile behaves in this kind of game.

One colour at a time: Suppose there is only one pile of counters, and each turn a player may remove between 1 and k of them (for some fixed k). As before, whoever takes the last counter wins.

Claim 1. *If it is your opponent's turn and the pile size is a multiple of $k+1$, then you are guaranteed to win.*

Why this works: Suppose the pile has $(k+1)$ counters and it is your opponent's turn. Whatever number r (where $1 \leq r \leq k$) your opponent removes, you respond by removing $(k+1)-r$ counters. This is possible since $1 \leq (k+1)-r \leq k$. Together, the two of you have removed $k+1$ counters in that round, and it is your opponent's turn again.

Now suppose instead the pile has $x(k+1)$ counters for some positive integer x. By repeating the above strategy, you can ensure that after each round, $k+1$ counters vanish from the pile. After x such rounds, all $x(k+1)$ counters are gone, and as you took the last counter, you win. ■

Applying this to our two colours separately:

- **White counters** (up to 3 removable per turn): a multiple of $3 + 1 = 4$ white counters, on your opponent's turn, guarantees your win.
- **Black counters** (up to 2 removable per turn): a multiple of $2 + 1 = 3$ black counters, on your opponent's turn, guarantees your win.

Combining both colours: If both piles are present, the two ideas above combine very naturally.

Claim 2. *If, at the start of your opponent's turn, the number of white counters is a multiple of* 4 and *the number of black counters is a multiple of* 3*, then you are guaranteed to win.*

Why this works: Whichever colour your opponent removes counters from, you respond with the pairing move for that colour. This keeps the other colour's count fixed (still a multiple of its magic number) while bringing the chosen colour down, 4 or 3 at a time, with your opponent moving first in each round and you moving second.

Eventually one colour runs out. At that point, whichever colour remains is a multiple of its magic number, and it is your opponent's turn – exactly the single-pile situation from before. From there you finish the job using the single-colour strategy. ■

We call a position where the white count is a multiple of 4 *and* the black count is a multiple of 3 a **loss case**: whoever is forced to move from a loss case is guaranteed to lose, if the opponent plays correctly.

3 Solving the Original Problem

We now return to the starting position: $w = 7$ white counters and $b = 3$ black counters, with Alice to move.

Note that $b = 3$ is already a multiple of 3, but $w = 7$ is not a multiple of 4. Rather, 7 leaves a remainder of 3 when divided by 4.

So, if Alice removes 3 white counters she leaves behind

$$w = 7 - 3 = 4, \qquad b = 3,$$

which is a loss case, and it is now Bob's turn. Alice is guaranteed to win from here. Therefore the answer is: **C (3 white counters).**

None of the other options reach a loss case in one move: removing 1 or 2 white counters leaves $w = 6$ or $w = 5$ (neither is a multiple of 4), and removing 1 or 2 black counters leaves $b = 2$ or $b = 1$ (neither is a multiple of 3, and $w = 7$ isn't a multiple of 4). So C is the unique correct first move.

4 Partial Generalization: Any Number of Counters

What strategy should we use if the bucket starts with w white counters and b black counters (arbitrary w and b) and the removal limits are the same (up to 3 white or up to 2 black per turn)?

> **Partially generalized problem.** A and B take turns removing counters from a bucket with w white and b black counters. The removal limits are up to 3 white or up to 2 black per turn. A moves first. How should A play to guarantee a win?

From Section 2, we know that a loss case is any position with $w \equiv 0 \pmod 4$ and $b \equiv 0 \pmod 3$. The strategy is therefore to steer the game towards the nearest loss case. We split into four cases based on the remainders of w and b.

Case 1: $w \equiv 0 \pmod 4$ **and** $b \equiv 0 \pmod 3$**.** The starting position is itself a loss case, and it is A's turn. So A is guaranteed to lose!

Case 2: $w \not\equiv 0 \pmod 4$ **and** $b \equiv 0 \pmod 3$**.** Let k be the remainder when w is divided by 4 (so $1 \leq k \leq 3$). If A removes k white counters, the position becomes $w - k \equiv 0 \pmod 4$ and $b \equiv 0 \pmod 3$: a loss case, with B to move. So A wins by taking k whites.

Case 3: $w \equiv 0 \pmod 4$ **and** $b \not\equiv 0 \pmod 3$**.** This is symmetric to Case 2. Let ℓ be the remainder when b is divided by 3. Removing ℓ black counters leaves a loss case for B. So A wins by taking ℓ blacks.

Case 4: $w \not\equiv 0 \pmod 4$ **and** $b \not\equiv 0 \pmod 3$**.** This is the interesting case, as neither remainder is 0. Let

$$i = w \bmod 4 \quad (1 \leq i \leq 3), \qquad j = b \bmod 3 \quad (1 \leq j \leq 2).$$

Definition 1 (Equalization)**.** *If a player's move brings the white and black remainders to the* same *value (i.e., makes $i = j$), we say that player has* ***equalized****.*

We now show that equal, non-zero remainders behave just like the all-zero loss case: whoever must move from a position with $i = j \neq 0$ is doomed to lose. The argument is the same as earlier; we use pairing, centred on i rather than 0 as the target remainder.

Claim 3. *If $i = j$, the player to move loses. If $i \neq j$, the player to move wins by equalizing in a single move.*

Why this works: If $i \neq j$, say $j > i$ (the other case is symmetric). Since $1 \leq j \leq 2$, the difference $j - i$ is at most 2, so it is a legal number of black counters to remove. Taking $j - i$ black counters brings the black remainder down to $j - (j - i) = i$, matching the white remainder. The position now has equal remainders, and it is the opponent's turn.

If $i = j$, the mover cannot avoid handing over an unequal (and hence losing) position. For, if the mover removes some white counters, changing the white remainder from i to a new value i'; since the amount removed is between 1 and 3, we get $i' \neq i$ within the cycle of remainders mod 4 (removing a non-zero amount always changes the remainder). So now $i' \neq j$, since $j = i \neq i'$. The position is unequal, and it is the opponent's turn – who can equalize by the argument above. The same reasoning applies if the mover instead removes black counters. Either way, the mover is forced to hand over an unequal position for the opponent to fix.

So whenever the remainders are equal, the mover is forced to break the equality, and the opponent is always able to restore it. Since the total number of counters decreases every turn, eventually the piles run out entirely, so both remainders hit 0 (the original loss case!), and it will be the mover's turn.

So the player who is stuck starting from $i = j$ ultimately loses. ■

Summarizing the winning strategy: This lets us describe the winning strategy for any starting position (w, b): compute $i = w \bmod 4$ and $j = b \bmod 3$. If $i = j$ (including $i = j = 0$), there is no winning first move.

Otherwise, remove $|i - j|$ counters from whichever colour has the larger remainder. You are then guaranteed to win.

Describing every outcome of Case 4 explicitly

We can now describe every outcome of Case 4 explicitly, in terms of i and j, where $i = w$ mod 4 and $j = b$ mod 3:

Remainders	**Winner**	**Winning move (if A wins)**
$i = 1,\ j = 1$	B	– (A has no winning move)
$i = 1,\ j = 2$	A	remove 1 black
$i = 2,\ j = 1$	A	remove 1 white
$i = 2,\ j = 2$	B	– (A has no winning move)
$i = 3,\ j = 1$	A	remove 2 whites
$i = 3,\ j = 2$	A	remove 1 white

Each of these follows directly from the claim that equal remainders ($i = j$) are losses for the mover, and unequal remainders are wins, achieved by removing the difference from the larger side.

5 Full Generalization: Any Removal Limits

Why stop at "up to 3 white or up to 2 black"? The same reasoning goes through unchanged if we allow up to m white counters or up to n black counters per turn, for any positive integers m and n.

Fully generalized problem. A and B take turns removing counters from a bucket with w white and b black counters.

> Each turn, a player removes between 1 and m white counters, or between 1 and n black counters. A moves first. How should A play to guarantee a win?

Every argument above goes through with 4 replaced by $m+1$ and 3 replaced by $n+1$.

Definition 2 (Generalized loss case)**.** *A position is a **generalized loss case** if $w \equiv 0 \pmod{m+1}$ and $b \equiv 0 \pmod{n+1}$. Whoever must move from such a position is guaranteed to lose.*

Let $i = w \bmod (m+1)$ and $j = b \bmod (n+1)$. As before, there are four cases:

Case 1 ($i = 0, j = 0$): The starting position is already a loss case, so A loses.

Case 2 ($i = 0, j \neq 0$): A removes j black counters, reaching a loss case for B.

Case 3 ($i \neq 0, j = 0$): A removes i white counters, reaching a loss case for B.

Case 4 ($i \neq 0, j \neq 0$): This splits further, as before:

- If $i = j$, the mover loses – there's no way to avoid breaking the equality, and the opponent will always restore it.
- If $i \neq j$, say (without loss of generality) $j > i$. Since $j \leq n$, the amount $j - i$ is a legal number of black counters to remove. Taking $j - i$ blacks equalizes the remainders and the mover wins.

The proofs of these facts are the same as in Section 4, with 4 and 3 replaced by $m+1$ and $n+1$.

This completes the solution to the fully general version of the bucket game: for any choice of starting counts w, b and removal limits m, n, the first player can immediately tell whether they are destined to win or lose, and exactly which counters to remove if they can win, just by comparing $w \bmod (m+1)$ and $b \bmod (n+1)$.

Comment: The key idea throughout was to find a "safe" combination of remainders that the opponent can never escape, and steer the game there as quickly as possible. Starting from single-colour pairing strategies, we built up to a complete solution for any starting numbers, and for any removal limits.

Acknowledgment

The authors would like to thank Shri Vinay Nair for posing the problem and urging us to work on it. The original problem is drawn from *Computational and Algorithmic Thinking 2011–2015, Book 2* by David I. Clark.

About the authors

AARAV LAD is a 13-year-old 8th grader from Bangalore. He is passionate about mathematics, playing the piano and cubing. His favourite topics include number theory and algebra. He has attended math camps such as Square Roots and Proofs. He can be reached at ladaarav@gmail.com.

AKSHOBHYAN R. is a 12-year-old 7th grader from Chennai. He is passionate about mathematics and likes to play chess. He has attended math

camps such as Epsilon India 2025. He can be reached at akshobhyan.rp@gmail.com.

HARI ANNAMALAI, a 14-year-old 10th grader from Bangalore. He is passionate about mathematics who primarily enjoys working on problems from number theory and mathematical games. He has attended math camps such as Epsilon India 2024, Epsilon India 2025 and PACT in the USA. He also loves playing chess. He can be reached at hari.k.annamalai@gmail.com.

NIDHI NAYAK is a 14-year-old homeschooler from Bangalore. She is passionate about mathematics and has attended math programs such as All Girls Math Nurture Camp, Raising A Mathematician Training Program and Math Path in the USA. She also enjoys swimming and crocheting. She can be reached at nidhijnayak@gmail.com.

SHAUNAK OKHDE is a 13-year-old homeschooler from Bangalore. He is passionate about mathematics, physics, piano and badminton and has attended math camps like Vedic Mathrix, Epsilon India 2024, Epsilon India 2025 and Cube Roots. He can be reached at shaunakokhde@gmail.com.