Primitive Pythagorean Triplets and Skandan Numbers

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Introduction

In this article we study Pythagorean triplets of the form $(x, x+1, z)$. Here are the first few examples of these triplets:

$$(3,4,5), \quad (20,21,29), \quad (119,120,169), \quad (696,697,985).$$

We see that the numbers increase rapidly. Are there easy ways to generate such triples? Are there an infinite number of them? In this article we describe a way to generate these triples, and this proves that there are an infinite number of them.

Brahmagupta's formula for generating PPTs

This is a formula for generating all Primitive Pythagorean Triplets (PPTs for short). If x, y, z are coprime positive integers such that $x^2 + y^2 = z^2$, then there exist coprime positive integers m, n (one odd and the other even) such that

$$x = m^2 - n^2, \quad y = 2mn, \quad z = m^2 + n^2, \tag{20}$$

OR

$$x = 2mn, \quad y = m^2 - n^2, \quad z = m^2 + n^2. \tag{21}$$

All PPTs can be expressed in this form. The triplets we are considering are primitive as x and $x + 1$ are necessarily coprime. Therefore, this is an appropriate formula to generate all such triplets.

Introducing Skandan's number

Generating PPTs of the form $(x, x+1, z)$ involves a special kind of number series. We shall refer to numbers in this series as *Skandan's numbers*. To see how they arise, we use the Brahmagupta formula.

If $x^2 + (x+1)^2 = z^2$, then there exist coprime positive integers m, n (one odd and the other even) such that

$$x = m^2 - n^2, \quad x + 1 = 2mn, \quad z = m^2 + n^2, \tag{22}$$

OR

$$x = 2mn, \quad x + 1 = m^2 - n^2, \quad z = m^2 + n^2. \tag{23}$$

In the first case we obtain, $m^2 - n^2 - 2mn = -1$,

$$\therefore\ n^2 + 2mn = m^2 + 1,$$

$$\therefore\ m^2 + 2mn + n^2 = 2m^2 + 1,$$

$$\therefore\ (m+n)^2 = 2m^2 + 1. \tag{24}$$

In the second case we obtain, $m^2 - n^2 - 2mn = 1$,

$$\therefore\ m^2 - 2mn = n^2 + 1,$$

$$\therefore\ m^2 - 2mn + n^2 = 2n^2 + 1,$$

$$\therefore\ (m-n)^2 = 2n^2 + 1. \tag{25}$$

We see that in both cases we get an equation of the form

$$S^2 = 1 + 2q^2 \tag{26}$$

where S and q are positive integers ($S = m + n$ in the 1st case, and $S = m - n$ in the 2nd case).

> **Claim:** If we get numbers S and q satisfying (26), we can generate two Pythagorean triplets of the form $(x, x+1, z)$.

Let's confirm.

Let S, q satisfy (26). Clearly, $S > q$.

We can now compute m, n in two ways:

- $m = q = \sqrt{(S^2-1)/2}, \quad n = S - m;$
- $n = q = \sqrt{(S^2-1)/2}, \quad m = S + n.$

The two (m, n) pairs now yield two PPTs via the Brahmagupta's formula.

Example One solution to (26) is $S = 3$, $q = 2$. These yield two (m, n) pairs:

- $m = 2,\ n = 1$, giving $m^2 - n^2 = 3$, $2mn = 4$, $m^2 + n^2 = 5$.

  We obtain the PPT $(3, 4, 5)$.

- $n = 2,\ m = 5$, giving $m^2 - n^2 = 21$, $2mn = 20$, $m^2 + n^2 = 29$.

  We obtain the PPT $(20, 21, 29)$.

Both PPTs are of the required form.

We may check algebraically that the triplets will always have the needed property (i.e., that the first two numbers differ by 1):

- In the first case we get

$$\begin{aligned} 2mn - (m^2 - n^2) &= m^2 + 2mn + n^2 - 2m^2 \\ &= (m+n)^2 - 2m^2 \\ &= S^2 - 2q^2 = 1. \end{aligned}$$

- In the second case we get

$$\begin{aligned} (m^2 - n^2) - 2mn &= m^2 - 2mn + n^2 - 2n^2 \\ &= (m-n)^2 - 2n^2 \\ &= S^2 - 2q^2 = 1. \end{aligned}$$

Hence, by finding a pair (S, q) with property (26), we are able to generate two PPTs with the needed property.

It follows that if we can generate infinitely many such pairs (S, q), then we can generate infinitely many PPTs of the type $(x, x+1, z)$.

Definition: A *Skandan number* is a positive integer $S > 1$ with the property that $\frac{1}{2}(S^2 - 1)$ is a perfect square.

For example, 3 and 17 have this property (they are the two smallest positive integers with the property):

$$\frac{3^2 - 1}{2} = 4 = 2^2, \qquad \frac{17^2 - 1}{2} = 144 = 12^2.$$

Generating Skandan Numbers

We now consider the problem of generating all Skandan numbers. Let S be a Skandan number, say

$$S^2 = 1 + 2q^2$$

for some positive integer q. Then $S^2 - 1 = 2q^2$, so,

$$(S - 1)(S + 1) = 2q^2. \tag{27}$$

Since $2q^2$ is even, $S^2 - 1$ is even, so S is odd, so both $S - 1$ and $S + 1$ are even. Indeed, they are consecutive even numbers; one of them is divisible by 4, while the other one is twice an odd number. This means that their GCD is 2; they do not share any prime factor other than 2.

There are thus two possible situations which yield a solution for (27):

(a) The first possibility is that $S + 1$ is a multiple of 4, and $S - 1$ is twice an odd number. In this case we may write

$$(S + 1) \cdot \left(\frac{S - 1}{2}\right) = q^2. \tag{28}$$

Since $S+1$ and $\frac{1}{2}(S-1)$ are coprime, (28) implies that both $S+1$ and $\frac{1}{2}(S-1)$ are perfect squares, say

$$S+1 = A^2, \quad \frac{S-1}{2} = B^2, \tag{29}$$

where A, B positive integers.

(b) The second possibility is that $S-1$ is a multiple of 4, and $S+1$ is twice an odd number. In this case we may write

$$(S-1) \cdot \left(\frac{S+1}{2}\right) = q^2. \tag{30}$$

Since $S-1$ and $\frac{1}{2}(S+1)$ are coprime, (28) implies that both $S-1$ and $\frac{1}{2}(S+1)$ are perfect squares, say

$$S-1 = A^2, \quad \frac{S+1}{2} = B^2, \tag{31}$$

where A, B positive integers.

Producing more Skandan numbers

From (31) there follows an interesting development.

Since $S-1 = A^2$ and $S+1 = 2B^2$, we must have

$$A^2 = 2B^2 - 2.$$

Hence A is even. Let $A = 2C$, where C is a positive integer. Then $A^2 = 4C^2$, giving $4C^2 = 2B^2 - 2$, hence $B^2 = 2C^2 + 1$.

But this means that B is a Skandan number.

From (31) we also get $S = 2B^2 - 1$. This tells us that it is possible to generate infinitely many Skandan numbers starting from any Skandan number, by iterating this formula repeatedly:

$$S_{\text{new}} = 2(S_{\text{old}})^2 - 1. \tag{32}$$

Thus we have:

$$\begin{aligned} 3 &\mapsto 2(3^2) - 1 = 17 \\ &\mapsto 2(17^2) - 1 = 577 \\ &\mapsto 2(577^2) - 1 = 665857 \\ &\mapsto 2(665857)^2 - 1 = 886731088897 \mapsto \cdots \end{aligned}$$

We may verify that 17, 577, 665857, 886731088897 are all Skandan numbers. Obviously, we can continue this progression for ever. This proves that there are infinitely many PPTs of the type $(x, x+1, z)$.

But note that the mechanism described above does *not* generate all possible Skandan numbers. For example, 99 is a Skandan number, but it is not captured by the above formula.

Examining the other possibility

Will the other possibility yield more Skandan numbers? From (29) we get, since $S + 1 = A^2$ and $S - 1 = 2B^2$,

$$A^2 = 2B^2 + 2. \tag{33}$$

Hence A is even. Let $A = 2C$, where C is a positive integer. If B were even, then the left side of (33) would be $0 \pmod 4$, as A is even, while the right side would be 2 (mod 4); not possible. Hence B is odd. Let $B = 2D + 1$, where D is a positive integer.

Now let us write (33) as $4C^2 - 4 = 2(2D+1)^2 - 2$. Simplifying, we get

$$(C-1)(C+1) = 2D(D+1). \tag{34}$$

As $2D(D+1)$ is even, C must be odd. Let $C = 2E + 1$. Substituting in (34) we get $2E(2E+2) = 2D(D+1)$, giving

$$D(D+1) = 2E(E+1). \tag{35}$$

Now consider the original context that gave rise to this exploration: PPTs of the form $(x, x+1, z)$. From the defining condition we get

$$x^2 + (x+1)^2 = z^2,$$

$$\therefore\ 2x^2 + 2x = z^2 - 1,$$

$$\therefore\ 2x(x+1) = (z-1)(z+1).$$

We see that z is odd; let $z = 2u + 1$. This yields $2x(x+1) = 4u(u+1)$, or

$$x(x+1) = 2u(u+1). \tag{36}$$

Observe the close similarity between (35) and (36). It tells us that from a PPT $(x, x+1, z)$. we should be able to produce a Skandan number, by computing the various quantities and tracing back! We only have to keep in mind the following connecting equalities:

$$B = 2D + 1 \quad \text{and} \quad A = 2(2E+1), \tag{37}$$

and

$$z = 2u + 1. \tag{38}$$

Examples

Start with the PPT $(3,4,5)$. Here $x = 3$ and $z = 5$, hence $u = 2$. So let's put $D = 3$ and $E = 2$ in (35). This yields, from (37),

$$B = 7, \quad A = 10,$$

and $S = 10^2 - 1 = 99$.

Similarly, from the PPT $(20,21,29)$ we get, in turn, $x = 20$, $z = 29$, $u = 14$, $D = 20$, $E = 14$, $B = 41$, $A = 58$, and

$$S = 58^2 - 1 = 3363.$$

We may verify that this is a Skandan number:

$$\frac{3363^2 - 1}{2} = 5654884 = 2378^2.$$

In general, from the PPT $(x, x+1, z)$ we obtain the Skandan number

$$S = \frac{z^2 - 1}{2}. \tag{39}$$

Acknowledgement

This article was motivated by a problem posed in a maths class (*Ganit Manthan*). Our teacher had got this problem when another student (Vivaan Jeedigunta) from a different batch asked,

> "We know a way to generate PPTs (a,b,c) where b and c are consecutive. Is there a way to generate PPTs where a and b are consecutive?"

About the author

Skandan Suresh studies in Grade 8 at Prakriti School Noida. He has a keen interest in mathematics and has been pursuing this by joining various maths camps. He also has an avid interest in music. He has been learning to play the flute in the Carnatic style and has performed at various opportunities including the G20 summit 2023. Skandan is a sports enthusiast too. He loves to read books and travel.

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